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Functional Differential Equations and Nonlinear Semigroups in L^p-Spaces

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The autonomous nonlinear functional differential equation $\dot{x}(t) = F(x_t)$, $t \geqslant 0$, $x_0 = \phi$ is studied as a semigroup of nonlinear operators in L^p function spaces. The method employed is to construct a semigroup of nonlinear operators which may be associated with the solutions of this equation. New existence and stability results are obtained for this equation by means of the semigroup approach.

1. INTRODUCTION

Our objective is to study the solutions to the autonomous nonlinear functional differential equation

$$\begin{aligned} x_0(\phi, h) &= \phi \in L^p(-r, 0; H), \qquad x(\phi, h)(0) = h \in H, \\ \dot{x}(\phi, h)(t) &= f(x(\phi, h)(t)) + F(x_t(\phi, h)), \qquad t \geqslant 0, \end{aligned} \tag{FDE}$$

as a semigroup of nonlinear operators. The notation of (FDE) means $r > 0$, $p \geqslant 1$, H is a Hilbert space, $x(\phi, h)(t)$: $[-r; \infty) \to H$, $x_t(\phi, h) \in L^p(-r, 0; H)$ is defined by $x_t(\phi, h)(\theta) = x(\phi, h)(t + \theta)$ for almost all $\theta \in [-r, 0]$, $f: H \to H$, and $F: L^p(-r, 0; H) \to H$.

Our approach will be to construct a generator for a nonlinear semigroup that may be associated with the solutions of (FDE) and then appeal to general results from nonlinear semigroup theory. We then have the problem of determining the exact sense in which the trajectories of this nonlinear semigroup provide solutions to (FDE). The setting of L^p function spaces allows for very general initial functions ϕ, as well as provides very tractable spaces for the study of the solutions. The treatment of (FDE) in L^p spaces has been undertaken by several authors, and some of these are listed in our references. Our work will continue the investigations of [14], where the underlying space was taken as a space of continuous functions as in the development of Hale [10]. By allowing H to be an arbitrary Hilbert space and f to be possibly unbounded in H, we can apply our results to partial differential equations with deviating arguments as in [14].

In the treatment of functional differential equations with L^p initial functions, it is necessary to take as the initial point the pair $\{\phi, h\} \in L^p(-r, 0; H) \times H$. This fact may be seen by comparing the solutions of the scalar delay differential equation

$$x_0(\phi, h) = \phi \in L^p(-1, 0; \mathbb{R}), \qquad x(\phi, h)(0) = h \in \mathbb{R},$$
$$x(\phi, h)(t) = x(\phi, h)(t-1), \qquad t \geqslant 0,$$

for $\phi_1 = 0$, $h_1 = 0$, $\phi_2 = 0$, and $h_2 = 1$. If this equation is formulated as in (FDE), then $r = 1$, $X = \mathbb{R}$, $f = 0$, and $F(\phi) = \phi(-1)$. The solution $x(\phi_1, h_1)(t)$ for $\{\phi_1, h_1\}$ is identically 0 on $[-1, \infty)$, whereas the solution for $\{\phi_2, h_2\}$ is given by $x(\phi_2, h_2)(t) = 0$ for $-1 \leqslant t < 0$ and $x(\phi_2, h_2)(t) = t^{n-1}/(n-1)!$ for $(n-1) \leqslant t < n$, $n = 1, 2,\ldots$. Accordingly, the nonlinear semigroup we construct for (FDE) will be defined in the Banach space $L^p(-r, 0; H) \times H$.

In Section 2, we set forth some necessary preliminaries from the theory of nonlinear accretive operators. In Section 3, we treat (FDE) for the class of F which are Lipschitz continuous from $L^p(-r, 0; H)$ to H. In Section 4, we treat (FDE) for a class of F which have a special form, but which are not necessarily continuous. This class of F is more difficult to handle than the continuous case, but is important because it applies to equations of delay type such as the example above. In order to apply the theory of nonlinear accretive operators for this class of F, we must use a weighted norm in $L^p(-r, 0; H)$. In Section 5, we discuss the connection between the nonlinear semigroup we construct and the solutions of (FDE).

2. Preliminaries

For a Banach space X the *duality mapping* $J: X \to 2^{X^*}$ is defined by

$$j \in J(x) \quad \text{iff } \langle x, j\rangle = \| x \|^2 = \| j \|^2 \quad \text{for} \quad x \in X. \tag{2.1}$$

The (nonlinear) operator $A: X \to X$ is *accretive* iff

$$\text{for all } x, y \in D(A) \quad \text{and some} \quad j \in J(x-y), \quad \langle Ax - Ay, j\rangle \geqslant 0. \tag{2.2}$$

By a result of Kato [12, Lemma 1.1], A is accretive iff

$$\|(I + \lambda A)\, x - (I + \lambda A)\, y \| \geqslant \| x - y \| \quad \text{for all } x, y \in D(A), \quad \lambda > 0. \tag{2.3}$$

By a result of Crandall and Liggett [7, Theorem I], if for some $\gamma \in \mathbb{R}$, $A + \gamma I: X \to X$ is accretive, $R(I + \lambda A)$ is onto for all sufficiently small $\lambda > 0$, and $D(A)$ is dense in X, then

$$\lim_{n\to\infty} (I + t/nA)^{-n}x \stackrel{\text{def}}{=} T(t)x \tag{2.4}$$

exists for all $x \in X$ and $t \geqslant 0$. Moreover, $T(t)$, $t \geqslant 0$ is a *strongly continuous semigroup of nonlinear operators on* X:

$$T(0)\, x = x \qquad \text{for all } x \in X; \tag{2.5}$$

$$T(t)\, x \text{ is continuous in } t \qquad \text{for each fixed } x \in X; \tag{2.6}$$

$$T(t_1 + t_2) = T(t_1)\, T(t_2), \qquad t_1, t_2 \geqslant 0; \tag{2.7}$$

$$\| T(t)\, x - T(t)\, y \| \leqslant e^{\gamma t} \| x - y \|, \qquad x, y \in X, \qquad t \geqslant 0. \tag{2.8}$$

Let H be a Hilbert space with norm $|\ |$ and inner product $(\ ,\ )$. Let $r > 0$ and let $1 \leqslant p < \infty$. Let τ: $[-r, 0] \to [0, \infty)$ be nondecreasing and bounded such that $\lim_{\theta\to -r} \tau(\theta) \neq 0$. We will treat (FDE) in the space $X = L^p(-r, 0; H; \mu) \times H$, where $d\mu(\theta) = \tau(\theta)\, d\theta$, with norm

$$\|\{\phi, h\}\| = \left(\int_{-r}^{0} |\phi(\theta)|^p\, \tau(\theta)\, d\theta + |h|^p\right)^{1/p}. \tag{2.9}$$

The duality mapping for X is given by the following.

PROPOSITION 2.1. *If $\{\phi, h\} \in X$, then $j \in J(\{\phi, h\})$ where j is defined by*

$$\textit{for all } \{\psi, k\} \in X, \qquad \langle\{\psi, k\}, j\rangle$$
$$= \|\{\phi, h\}\|^{2-p} \left(\int_{-r}^{0} (\psi, \phi)\, |\phi|^{p-2} \tau\, d\theta + (k, h)\, |h|^{p-2}\right). \tag{2.10}$$

Proof. For all $\psi \in L^p(-r, 0; H; \mu)$, $\int_{-r}^0 (\psi, \phi)|\phi|^{p-2}\, \tau\, d\theta$ exists because $|\phi|^{(p-1)q} = |\phi|^p$ implies that $|\phi|^{p-1} \in L^q(-r, 0; H; \mu)$, $1/p + 1/q = 1$. Obviously, $\langle\{\phi, h\}, j\rangle = \|\{\phi, h\}\|^2$. We have only to show that $|\langle\{\psi, k\}, j\rangle| \leqslant \|\{\psi, k\}\|\, \|\{\phi, h\}\|$. By the Cauchy–Schwartz inequality,

$$|\langle\{\psi, k\}, j\rangle| \leqslant \|\{\phi, h\}\|^{2-p} \left(\left(\int_{-r}^{0} |\psi|^p \tau\right)^{1/p} \left(\int_{-r}^{0} |\phi|^{(p-1)q} \tau\right)^{1/q} + |k|\, |h|^{p-1}\right).$$

Set $a = \int_{-r}^0 |\phi|^p\, \tau$, $b = \int_{-r}^0 |\psi|^p\, \tau$, $c = |h|$, and $d = |k|$, and it suffices to show that $(a + c)^{(2-p)/p}(b^{1/p}a^{1/q} + dc^{p-1}) \leqslant (a + c^p)^{1/p}(b + d^p)^{1/p}$, or $b^{1/p}a^{1/q} + dc^{p-1} \leqslant (a + c^p)^{1/q}(b + d^p)^{1/p}$. But this follows from the inequality $a_1b_1 + a_2b_2 \leqslant (a_1{}^p + a_2{}^p)^{1/p}(b_1{}^q + b_2{}^q)^{1/q}$ with $a_1 = b^{1/p}$, $b_1 = a^{1/q}$, $a_2 = d$, and $b_2 = c^{p-1}$.

3. The Equation (FDE) with F Lipschitz Continuous

In this section, we treat (FDE) in the space X with $\tau(\theta) \equiv 1$ in (2.9). We require that $f\colon H \to H$ is a densely defined (nonlinear) operator such that for some $\alpha \in \mathbb{R}$, $-f + \alpha I$ is accretive in H, and $R(I - \lambda f) = H$ for $\lambda > 0$ sufficiently small. Notice that f need not be continuous. We require also that $F\colon L^p(-r, 0; H; \mu) \to H$ is everywhere defined and Lipschitz continuous with Lipschitz constant β. Define $A\colon X \to X$ by

$$\begin{aligned} D(A) &= \{\{\phi, h\} \in X\colon \phi \text{ is absolutely continuous on } [-r, 0], \\ &\quad \phi' \in L^p(-r, 0; H; \mu), \text{ and } h = \phi(0) \in D(f)\} \\ A\{\phi, h\} &= \{-\phi', -f(\phi(0)) - F(\phi)\}. \end{aligned} \tag{3.1}$$

Obviously, A is densely defined. We will show that A is the generator of a nonlinear semigroup in the sense of Eq. (2.4).

Proposition 3.1. *$A + \gamma I$ is accretive in X where $\gamma = \max\{0, 1/p + \alpha\} + \beta$.*

Proof. Let $\{\phi_1, \phi_1(0)\}$, $\{\phi_2, \phi_2(0)\} \in D(A)$, let $\phi = \phi_1 - \phi_2$, and let $j \in J(\{\phi, \phi(0)\})$ as in (2.10). Then,

$$\begin{aligned} &\langle A\{\phi_1, \phi_1(0)\} - A\{\phi_2, \phi_2(0)\}, j\rangle \\ &= \|\{\phi, \phi(0)\}\|^{2-p} \Big(-\int_{-r}^0 (\phi', \phi)\,|\phi|^{p-2} \\ &\qquad - (f(\phi_1(0)) + F(\phi_1) - f(\phi_2(0)) - F(\phi_2), \phi(0))\,|\phi(0)|^{p-2}\Big) \\ &= \|\{\phi, \phi(0)\}\|^{2-p} \Big(-\int_{-r}^0 1/p\; d/d\theta\, |\phi(\theta)|^p\, d\theta \\ &\qquad - (f(\phi_1(0)) - f(\phi_2(0)), \phi(0))\,|\phi(0)|^{p-2} \\ &\qquad - (F(\phi_1) - F(\phi_2), \phi(0))\,|\phi(0)|^{p-2}\Big) \\ &\geqslant \|\{\phi, \phi(0)\}\|^{2-p} (-1/p(|\phi(0)|^p - |\phi(-r)|^p) \\ &\qquad - \alpha\,|\phi(0)|^p - \beta\,\|\{\phi, \phi(0)\}\|\,|\phi(0)|^{p-1}) \\ &\geqslant -\gamma\,\|\{\phi, \phi(0)\}\|^2 \end{aligned} \tag{3.2}$$

and this yields the accretiveness of $A + \gamma I$.

Proposition 3.2. *$(R(I + \lambda A) = X$ for sufficiently small $\lambda > 0$. If H is finite dimensional then $(I + \lambda A)^{-1}$ is compact for sufficiently small $\lambda > 0$.*

Proof. Let λ be positive and sufficiently small and let $\{\psi, k\} \in X$. Define the mapping $d: H \to H$ by

$$d(h) = (I - \lambda f)^{-1} \left(k + \lambda F\left(e^{\theta/\lambda} h + (e^{\theta/\lambda}/\lambda) \int_\theta^0 e^{-s/\lambda} \psi(s)\, ds\right)\right).$$

Then d is a strict contraction on H, since

$$| d(h_1) - d(h_2)| \leqslant (\lambda\beta/1 - \lambda\alpha)\, |h_1 - h_2|.$$

Thus, d has a unique fixed point h_0. Define

$$\phi(\theta) \stackrel{\text{def}}{=} e^{\theta/\lambda} h_0 + (e^{\theta/\lambda}/\lambda) \int_\theta^0 e^{-s/\lambda} \psi(s)\, ds.$$

Then, ϕ is absolutely continuous on $[-r, 0]$ and $\phi - \lambda\phi' = \psi$ a.e. on $[-r, 0]$. Also, $\phi(0) = h_0 \in D(f)$, since $h_0 = (I - \lambda f)^{-1}(k + \lambda F(\phi))$. Therefore, $(I + \lambda A)\{\phi, \phi(0)\} = \{\psi, k\}$. By (2.3) $(I + \lambda A)^{-1}$ exists and is Lipschitz continuous on X. This fact and the fact that $\psi \to \int_\theta^0 e^{-s/\lambda}\psi(s)\, ds$ is compact in $L^p(-r, 0; H)$ if H is finite dimensional [11, p. 157] yield the second claim.

By virtue of Propositions 3.1 and 3.2 the nonlinear operator A satisfies the hypothesis of Theorem I of M. Crandall and T. Liggett [7], and therefore, generates a nonlinear semigroup $T(t)$, $t \geqslant 0$ as in (2.4). Notice that γ is necessarily positive in (2.8).

4. The Equation (FDE) with F Not Necessarily Lipschitz Continuous

In this section we will suppose that the function F in (FDE) has a special form, but is not necessarily Lipschitz continuous. Let $\eta: [-r, 0] \to \text{Lip}(H, H)$, where Lip (H, H) is the Banach space of Lipschitz continuous operators on H with norm

$$| g |_{\text{Lip}(H,H)} = \sup_{x,y \in H} | g(x) - g(y)|/| x - y | + | g(0)|, \qquad g \in \text{Lip}(H, H).$$

We require that η is of bounded variation on $[-r, 0]$, $\eta(-r) = 0$, and $\lim_{\theta \to -r} \eta(\theta) \neq 0$. Let $\tau(\theta) = \int_{-r}^\theta | d\eta |$, the total variation of η between $-r$ and θ. We will investigate (FDE) in the space X with the norm weighted by this τ as in (2.9). Let $F: C(-r, 0; H) \to H$ by $F(\phi) = g(\int_{-r}^0 d\eta(\theta)\, \phi(\theta))$, $g: H \to H$ is Lipschitz continuous with Lipschitz constant β. Notice that F is Lipschitz continuous from $C(-r, 0; H)$ to H, but not necessarily Lipschitz continuous from $L^p(-r, 0; H; \mu)$ to H (in fact, not even well defined). Let $f: H \to H$ be densely defined (nonlinear) such that for some $\alpha \in \mathbb{R}$,

$-f+\alpha I$ is accretive in H and $R(I-\lambda f)=H$ for all sufficiently small $\lambda>0$. Define $A\colon X\to X$ by

$$\begin{aligned} D(A) = \{\{\phi, h\} \in X\colon &\phi \text{ is absolutely continuous on } [-r,0],\\ &\phi' \in L^p(-r,0;H;\mu), \text{ and } h=\phi(0)\in D(f)\},\\ &A\{\phi,h\} = \{-\phi', -f(\phi(0)) - F(\phi)\}. \end{aligned} \tag{4.1}$$

Obviously, A is densely defined in X. The necessity of weighting the norm with τ arises in order to show the accretiveness condition on A. If this function τ is absent, then the semigroup $T(t)$, $t\geqslant 0$, will in general satisfy a condition of the form $| T(t)|_{\mathrm{Lip}(H,H)} \leqslant Me^{\omega t}$, where $M\geqslant 1$. We will require the following lemma.

LEMMA 4.1. *Let $u, v\colon [-r,0]\to\mathbb{R}$ such that u is absolutely continuous on $[-r,0]$ and v is of bounded variation on $[-r,0]$. Then, $\int_{-r}^0 u'(\theta)\, v(\theta)\, d\theta = -\int_{-r}^0 u(\theta)\ dv(\theta) + u(0)\, v(0) - u(-r)\, v(-r)$.*

Proof. Since u is continuous and v is of bounded variation the Stieltjes integral $\int_{-r}^0 u(\theta)\, dv(\theta)$ exists and

$$\int_{-r}^0 u(\theta)\, dv(\theta) = u(0)\, v(0) - u(-r)\, v(-r) - \int_{-r}^0 v(\theta)\, du(\theta).$$

From [13, Corollary 56.5, p. 311], we have that $\int_{-r}^0 v(\theta)\, du(\theta) = \int_{-r}^0 u'(\theta)\, v(\theta)\, d\theta$ and the conclusion follows.

PROPOSITION 4.1. *If $p>1$, then $A+\gamma I$ is accretive in X where $\gamma=\max\{0, \tau(0)(1/p+\beta^q/q)+\alpha\}$, $1/p+1/q=1$. If $p=1$ and $\beta\leqslant 1$, then $A+\gamma I$ is accretive in X where $\gamma=\max\{0,\tau(0)+\alpha\}$.*

Proof. Let $\{\phi_1, \phi_1(0)\}$, $\{\phi_2,\phi_2(0)\}\in D(A)$, let $\phi=\phi_1-\phi_2$, and let $j\in J(\{\phi,\phi(0)\})$ as in (2.10). As in (3.2) we have, using Lemma 4.1,

$$\begin{aligned} &\langle A\{\phi_1,\phi_1(0)\} - A\{\phi_2,\phi_2(0)\}, j\rangle\\ &\quad = \|\{\phi,\phi(0)\}\|^{2-p} \Big(-\int_{-r}^0 1/p\ d/d\theta\ |\phi(\theta)|^p\, \tau(\theta)\, d\theta\\ &\qquad - (f(\phi_1(0)) - f(\phi_2(0)), \phi(0))\ |\phi(0)|^{p-2}\\ &\qquad - \Big(g\Big(\int_{-r}^0 d\eta(\theta)\,\phi_1(\theta)\Big) - g\Big(\int_{-r}^0 d\eta(\theta)\,\phi_2(\theta)\Big), \phi(0)\ |\phi(0)|^{p-2}\Big)\Big)\\ &\quad \geqslant \|\{\phi,\phi(0)\}\|^{2-p}\Big(\int_{-r}^0 1/p\ |\phi(\theta)|^p\, d\tau(\theta) - (1/p\ \tau(0)+\alpha)\ |\phi(0)|^p\\ &\qquad - \beta\int_{-r}^0 |\phi(\theta)|\ |\phi(0)|^{p-1}\, d\tau(\theta)\Big). \end{aligned} \tag{4.2}$$

If $p > 1$, then (4.2) $\geqslant \|\{\phi, \phi(0)\}\|^{2-p}(\int_{-r}^{0} k(|\phi(\theta)|)\, d\tau(\theta) - \gamma\, |\phi(0)|^p)$ where $k(x) = (1/p)x^p + (\beta^q/q)\, c^p - \beta x c^{p-1}$, $c = |\phi(0)|$. But $k(x) \geqslant 0$ for all $x \geqslant 0$, since $k(0) \geqslant 0$, $\lim_{x\to\infty} k(x) = \infty$, $k'(x_0) = 0$ iff $x_0 = c\beta^{1/p-1}$, and $k(x_0) = 0$. Therefore, using the fact that $\gamma \geqslant 0$, we have

$$\langle A\{\phi_1, \phi_1(0)\} - A\{\phi_2, \phi_2(0)\}, j\rangle \geqslant -\gamma\, \|\{\phi, \phi(0)\}\|^2.$$

If $p = 1$ and $\beta \leqslant 1$ then (4.2) $\geqslant$

$$\|\{\phi, \phi(0)\}\| \left(\int_{-r}^{0} (1 - \beta)\, |\phi(\theta)|\, d\tau(\theta) - (\tau(0) + \alpha)\, |\phi(0)|\right)$$
$$\geqslant -(\tau(0) + \alpha)\, \|\{\phi, \phi(0)\}\|\, |\phi(0)| \geqslant -\gamma\, \|\{\phi, \phi(0)\}\|^2.$$

Thus, $A + \gamma I$ is accretive in both cases and the proof is finished.

We remark that when $p > 1$, the quantity $1/p + \beta^q/q \geqslant \beta$. To see this define $h(\beta) = 1/p + \beta^q/q - \beta$, $\beta \geqslant 0$ and observe that $h'(\beta_0) = 0$ iff $\beta_0 = 1$ and $h(1) = 0$.

PROPOSITION 4.2. *$R(I + \lambda A) = X$ for sufficiently small $\lambda > 0$. If H is finite dimensional then $(I + \lambda A)^{-1}$ is compact for sufficiently small $\lambda > 0$.*

Proof. Let $\lambda > 0$ be sufficiently small and let $\{\psi, k\} \in X$. Define the mapping $d\colon H \to H$ by

$$d(h) = (I - \lambda f)^{-1} \left(k + \lambda g \left(\int_{-r}^{0} \left(e^{\theta/\lambda} h + (e^{\theta/\lambda}/\lambda) \int_{\theta}^{0} e^{-s/\lambda} \psi(s)\, ds\right) d\eta(\theta)\right)\right).$$

For sufficiently small $\lambda > 0$, d is a strict contraction on H, since

$$|d(h_1) - d(h_2)| \leqslant (\lambda\beta/1 + \lambda\alpha)\, |h_1 - h_2|\, \tau(0).$$

Let h_0 be the unique fixed point of d and define

$$\phi(\theta) \stackrel{\text{def}}{=} e^{\theta/\lambda} h_0 + (e^{\theta/\lambda}/\lambda) \int_{\theta}^{0} e^{-s/\lambda} \psi(s)\, ds.$$

Then ϕ is absolutely continuous on $[-r, 0]$, $\phi - \lambda\phi' = \psi$ a.e. on $[-r, 0]$, $\phi(0) = h_0 \in D(f)$, and therefore, $(I + \lambda A)\, \phi = \psi$. The second claim is proved exactly as in Proposition 3.2.

By virtue of Propositions 4.1 and 4.2, the nonlinear operator A (with $\beta \leqslant 1$ if $p = 1$) satisfies the hypothesis of Theorem I of Crandall and Liggett [7], and therefore, generates a nonlinear semigroup $T(t)$, $t \geqslant 0$ in the sense of (2.4). Notice that if $\gamma = 0$ in (2.8), then the trajectories are stable in the norm of X as $t \to \infty$. In this case $\alpha < 0$, and this means that the ordinary part of (FDE) corresponding to f serves as a damping term for the equation.

5. The Correspondence of $T(t)$, $t \geqslant 0$ to the Solutions of (FDE)

In certain cases, the trajectories of the semigroup $T(t)$, $t \geqslant 0$, of Sections 3 and 4 can be shown to give solutions to the equation (FDE). Denote by P_1 the projection of X onto $L^p(-r, 0; H; \mu)$ and by P_2 the projection of X onto H (that is, $P_1\{\phi, h\} = \phi$ and $P_2\{\phi, h\} = h$). Define for $\{\phi, h\} \in X$ the function $x(\phi, h)(t)$: $[-r, \infty) \to H$ by

$$x(\phi, h)(t) = \phi(t) \qquad \text{if} \quad -r \leqslant t < 0,$$

and

$$x(\phi, h)(t) = P_2 T(t)\{\phi, h\} \qquad \text{if} \quad t \geqslant 0. \tag{5.1}$$

We observe that $x(\phi, h)(t)$ is continuous in t for $t \geqslant 0$ by virtue of (2.6), the strong continuity of $T(t)$, $t \geqslant 0$.

In order to show that the function defined in (5.1) is, at least in certain cases, a solution of (FDE), we will demonstrate a relationship between $T(t)$, $t \geqslant 0$ and the semigroup associated with (FDE) in $C(-r, 0; H)$. Specifically, let $C = C(-r, 0; H)$ and define $Q: X \to C$ by $D(Q) = \{\{\phi, h\} \in X: \phi$ agrees a.e. on $[-r, 0]$ with a continuous function ψ and $\psi(0) = h\}$, $Q\{\phi, h\} = \psi$ (without confusion, we will sometimes write $Q\{\phi, h\} = \phi$, where ϕ means the continuous function identified with $\phi \in L^p(-r, 0; H; \mu)$. Note that $D(A) \subset D(Q)$, Q is one-to-one, and $Q^{-1}: C \to X$ is continuous by virtue of

$$\|\{\phi, h\}\|_X \leqslant \left(\int_{-r}^{0} \tau(\theta)\, d\theta + 1\right)^{1/p} \|Q\{\phi, h\}\|_C \qquad \text{for all } \{\phi, h\} \in D(Q). \tag{5.2}$$

Define $A_C : C \to C$ by $D(A_C) = \{\phi \in C: \phi' \in C, \phi(0) \in D(f)$, and $\phi'(0) = f(\phi(0)) + F(\phi)\}$, $A_C\phi = -\phi'$. Here we take F as in Sections 3 and 4, but restricted to C. We note that in Section 3, F restricted to C is Lipschitz continuous by virtue of the fact that $\|\phi\|_{L^p(-r,0:H;\mu)} \leqslant r^{1/p} \|\phi\|_C\, \tau(0)^{1/p}$ for all $\phi \in C$. In Section 4, F is also Liptschitz continuous on C. Denote by γ_C the Lipschitz constant of F on C. It is shown in [17] that $D(A_C)$ is dense in C, $A_C + \gamma_C I$ is accretive in C, $R(I + \lambda A_C) = C$ for all sufficiently small $\lambda > 0$, and A_C is the generator of a strongly continuous nonlinear semigroup $T_C(t)$, $t \geqslant 0$ on C by means of Eq. (2.4). Observe that

$$QA\{\phi, h\} = A_C Q\{\phi, h\} \qquad \text{if} \quad Q\{\phi, h\} \in D(A_C). \tag{5.3}$$

The following proposition says, roughly speaking, that $T(t)$, $t \geqslant 0$ and $T_C(t)$, $t \geqslant 0$, agree on $D(Q)$.

Proposition 5.1. *If $\{\phi, h\} \in D(Q)$, then $QT(t)\{\phi, h\} = T_C(t)\, Q\{\phi, h\}$ for all $t \geqslant 0$.*

Proof. Let $\phi \in C$ and $\lambda > 0$ sufficiently small. There exists $\{\psi, k\} \in D(A)$ such that $Q\{\psi, k\} \in D(A_C)$ and $(I + \lambda A_C) Q\{\psi, k\} = \phi$. From (5.3) we have

$$Q(I + \lambda A)\{\psi, k\} = (I + \lambda A_C) Q\{\phi, k\} = \phi. \tag{5.4}$$

Then (5.4) implies

$$(I + \lambda A_C)^{-1} \phi = Q(I + \lambda A)^{-1} Q^{-1}\phi \qquad \text{for all } \phi \in C. \tag{5.5}$$

For $\phi \in C$, $t > 0$, n a positive integer sufficiently large, (5.5) implies

$$(I + t/nA_C)^{-n} \phi = Q(I + t/nA)^{-n} Q^{-1}\phi. \tag{5.6}$$

Therefore, (5.6), (2.4), and (5.2) yield

$$\begin{aligned} &\| T(t)Q^{-1}\phi - Q^{-1}T_C(t)\phi \|_X \\ &= \lim_{n\to\infty} \|(I + t/nA)^{-n} Q^{-1}\phi - Q^{-1}T_C(t)\phi \|_X \\ &\leqslant \left(\int_{-r}^{0} \tau(\theta)\, d\theta + 1\right)^{1/p} \lim_{n\to\infty} \| Q(I + t/nA)^{-n} Q^{-1}\phi - T_C(t)\phi \|_C \\ &= \left(\int_{-r}^{0} \tau(\theta)\, d\theta + 1\right)^{1/p} \lim_{n\to\infty} \|(I + t/nA_C)^{-n} \phi - T_C(t)\phi \|_C = 0 \end{aligned}$$

and the proof is complete.

Define for $\phi \in C$ the function $y(\phi)(t)$: $[-r, \infty) \to H$ by

$$\begin{aligned} y(\phi)(t) &= \phi(t) \qquad \text{if} \quad -r \leqslant t \leqslant 0, \\ \text{and} \qquad y(\phi)(t) &= (T_C(t)\phi)(0) \qquad \text{if} \quad t \geqslant 0. \end{aligned} \tag{5.7}$$

The following important property of $T_C(t)$, $t \geqslant 0$ is proved by Flaschka and Leitman [9]:

$$y_t(\phi) = T_C(t)\,\phi \qquad \text{for all } t \geqslant 0, \quad \phi \in C. \tag{5.8}$$

PROPOSITION 5.2. *If* $\{\phi, h\} \in D(Q)$, *then*

$$x(\phi, h)(t) = y(Q\{\phi, h\})(t) \qquad \text{for} \quad t \geqslant -r. \tag{5.9}$$

Proof. For $\{\phi, h\} \in D(Q)$, $t \geqslant 0$, Eq. (5.1), Proposition 5.1, and Eq. (5.8) imply

$$\begin{aligned} x(\phi, h)(t) &= P_2 T(t)\{\phi, h\} = P_2 Q^{-1} T_C(t)\, Q\{\phi, h\} \\ &= (T_C(t)\, Q\{\phi, h\})(0) = y(Q\{\phi, h\})(t). \end{aligned}$$

Also, by definition, $x(\phi, h)(t) = y(Q\{\phi, h\})(t) = \phi(t)$ for $-r \leqslant t \leqslant 0$.

PROPOSITION 5.3. *If* $\{\phi, h\} \in X$, *then* $x_t(\phi, h) = P_1 T(t)\{\phi, h\}$ *for all* $t \geqslant 0$.

Proof. Let $\{\phi, h\} \in D(Q)$, $t \geqslant 0$. By (5.8), (5.9), and Proposition 5.1, we have

$$x_t(\phi, h) = y_t(Q\{\phi, h\}) = T_C(t)\, Q\{\phi, h\} = P_1 Q^{-1} T_C(t)\, Q\{\phi, h\} = P_1 T(t)\{\phi, h\}.$$

Now let $\{\phi, h\} \in X$, $t \geqslant 0$, and let $\{\phi_n, h_n\} \subset D(Q)$ converging in X to $\{\phi, h\}$. Then

$$\begin{aligned}
&\Big(\int_{r}^{0} |(P_1 T(t)\{\phi, h\})(\theta) - x(\phi, h)(t+\theta)|^p\, \tau(\theta)\, d\theta \\
&\quad \leqslant \Big(\int_{-r}^{0} |(P_1 T(t)\{\phi, h\})(\theta) - (P_1 T(t)\{\phi_n, \phi_n(0)\})(\theta)|^p\, \tau(\theta)\, d\theta \\
&\qquad + \Big(\int_{-r}^{0} |(P_1 T(t)\{\phi_n, \phi_n(0)\})(\theta) - x(\phi_n, \phi_n(0))(t+\theta)|^p\, \tau(\theta)\, d\theta \\
&\qquad + \Big(\int_{-r}^{-t} |\phi_n(t+\theta) - \phi(t+\theta)|^p\, \tau\theta\, d\theta \\
&\qquad + \int_{-t}^{0} |P_2 T(t+\theta)\{\phi_n, \phi_n(0)\} - P_2 T(t+\theta)\{\phi, h\}|^p\, \tau(\theta)\, d\theta \\
&\quad = (1) + (2) + (3).
\end{aligned} \tag{5.10}$$

As $n \to \infty$, $(1) \to 0$ by virtue of (2.8). Also, $(2) = 0$ by the fact established above. The first term in (3) converges to 0 as $n \to \infty$ because $\phi_n \to \phi$ in $L^p(-r, 0; H; \mu)$. The second term in (3) is bounded by $\|\{\phi_n, \phi_n(0)\} - \{\phi, h\}\|^p \int_{-t}^{0} e^{\gamma(t+\theta)p} \tau(\theta)\, d\theta$ by virtue of (2.8), and it too, converges to 0 as $n \to \infty$. Therefore, $(5.10) \to 0$, and the conclusion follows.

The next propostion says, roughly speaking, that the "translated segment" of the solution to (FDE) corresponding to $\{\phi, h\} \in X$ lies in C for $t \geqslant r$.

PROPOSITION 5.4. *If* $t \geqslant r$ *and* $\{\phi, h\} \in X$, *then* $T(t)\{\phi, h\} \in D(Q)$ *and* $QT(t)\{\phi, h\} = T_C(t-r)\, QT(r)\{\phi, h\}$.

Proof. By Proposition 5.3, $x_t(\phi, h) = P_1 T(t)\{\phi, h\}$ for $\{\phi, h\} \in X$, $t \geqslant 0$. By (5.1), $x(\phi, h)(t) = P_2 T(t)\{\phi, h\}$ if $t \geqslant 0$. By the strong continuity (2.6) of $T(t)$, $t \geqslant 0$, $P_2 T(t)\{\phi, h\}$ is continuous in t for $t \geqslant 0$. Thus, $x_t(\phi, h) \in C$ if $t \geqslant r$, and so $P_1 T(t)\{\phi, h\}$ agrees with a continuous function ψ a.e. on $[-r, 0]$ for these t. Also, $\psi(0) = x_t(\phi, h)(0) = x(\phi, h)(t) = P_2 T(t)\{\phi, h\}$. Hence, $T(t)\{\phi, h\} \in D(Q)$. Lastly, if $\{\phi, h\} \in X$ and $t \geqslant r$, we have by (2.7) and Proposition 5.1,

$$QT(t)\{\phi, h\} = QT(t-r)\, T(r)\{\phi, h\} = T_C(t-r)\, QT(r)\{\phi, h\}.$$

PROPOSITION 5.5. *If $t \geqslant r$ then for all $\{\phi, h\}, \{\psi, k\} \in X$,*

$$\| QT(t)\{\phi, h\} - QT(t)\{\psi, k\}\|_C \leqslant e^{\gamma t} \|\{\phi, h\} - \{\psi, k\}\|_X .$$

Consequently, $QT(t)$ maps bounded sets of X into bounded sets of C.

Proof. For $t \geqslant r$, $T(t)\{\phi, h\}$, $T(t)\{\psi, k\} \in D(Q)$ by Proposition 5.4. For a.e. $\theta \in [-r, 0]$, Proposition 5.3 yields

$$\begin{aligned}(QT(t)\{\phi, h\})(\theta) &= (P_1T(t)\{\phi, h)(\theta) = (x_t(\phi, h))(\theta)\\ &= x(\phi, h)(t+\theta) = P_2T(t+\theta)\{\phi, h\} \qquad (\text{since } t+\theta \geqslant 0).\end{aligned}$$

Thus,

$$\begin{aligned}&\sup_{\theta\in[-r,0]} |(QT(t)\{\phi, h\})(\theta) - (QT(t)\{\psi, k\})(\theta)|_H\\ &\quad = \sup_{\theta\in[-r,0]} | P_2T(t+\theta)\{\phi, h\} - P_2T(t+\theta)\{\psi, k\}|_H\\ &\quad \leqslant \sup_{\theta\in[-r,0]} \| T(t+\theta)\{\phi, h\} - T(t+\theta)\{\psi, k\}\|_X\\ &\quad \leqslant e^{\gamma t} \|\{\phi, h\} - \{\psi, k\}\|_X .\end{aligned}$$

Propositions 5.4 and 5.5 provide information about the properties of $T(t)$, $t \geqslant 0$ from the corresponding properties of $T_C(t)$, $t \geqslant 0$. For example, stability properties as $t \to \infty$ of $T_C(t)$, $t \geqslant 0$ in C, hold true for $T(t)$, $t \geqslant 0$ in X. Some other examples are provided by the following propositions.

PROPOSITION 5.6. *If $T_C(t)$ is compact from C to C for $t \geqslant r$, then $QT(t)$ is compact from X to C for $t \geqslant 2r$, and $T(t)$ is compact from X to X for $t \geqslant 2r$.*

Proof. The first claim is immediate from Propositions 5.4 and 5.5. The second claim follows from the first, the continuity of Q^{-1}, and the identity $T(t) = Q^{-1}QT(t)$.

PROPOSITION 5.7. *If $T_C(t)(C) \subset D(A_C)$ for $t \geqslant r$, then $QT(t)(X) \subset D(A_C)$ for $t \geqslant 2r$ and $T(t)(X) \subset D(A)$ for $t \geqslant 2r$.*

Proof. The first claim is immediate from Proposition 5.4. The second claim follows from the first and Eq. (5.3).

In general, the function defined in (5.1) may be thought of as a "generalized solution" of (FDE) with "translated segments" $P_1T(t)\{\phi, h\} \in L^p(-r, 0; H; \mu)$. The following two propositions give sufficient conditions for the function defined in (5.1) to satisfy the equation (FDE).

PROPOSITION 5.8. *Suppose either $p > 1$ of F is linear. If $\{\phi, h\} \in D(A)$ then*

$$\dot{x}(\phi, h)(t) = f(x(\phi, h)(t)) + F(x_t(\phi, h)) \qquad \textit{for a.e. } t \geqslant 0. \qquad (5.11)$$

Proof. If $p > 1$, then X and X^* are uniformly convex. From the general theory of nonlinear semigroups [12] we know that if $\{\phi, h\} \in D(A)$, then

$$d/dt\, T(t)\{\phi, h\} = -AT(t)\{\phi, h\} \qquad \text{for a.e. } t \geqslant 0. \tag{5.12}$$

If $p = 1$ and F is linear, then (5.12) holds for $t \geqslant 0$. (See [11, p. 481].) Then (5.11) follows from Proposition 5.3 by applying P_2 to both sides of (5.12) and using the definition of A.

PROPOSITION 5.9. *Let $p \geqslant 1$, let $f\colon H \to H$ be everywhere defined and continuous, and let $F\colon L^p(-r, 0; H; \mu) \to H$ be everywhere defined and Lipschitz continuous as in Section 3. If $\{\phi, h\} \in X$ then*

$$\dot{x}(\phi, h)(t) = f(x(\phi, h)(t)) + F(x_t(\phi, h)) \qquad \textit{for} \quad t \geqslant 0. \tag{5.13}$$

Proof. We will decompose $A = A_1 + A_2$, where A_1 is linear, $-A_1$ is the infinitesimal generator of a strongly continuous linear semigroup, and A_2 is nonlinear, but continuous and everywhere defined. We will then apply the results of [16] to this semilinear case. Define $A_1, A_2 : X \to X$ by

$$A_1\{\phi, h\} = \{-\phi', 0\}, \qquad D(A_1) = D(A),$$
$$A_2\{\phi, h\} = \{0, -f(h) - F(\phi)\}, \qquad D(A_2) = X.$$

Then $A = A_1 + A_2$ satisfies the hypothesis of [16, Theorem II], where the linear semigroup $S(t)$, $t \geqslant 0$ generated by $-A_1$ has the property $P_2S(t)\{\phi, h\} = P_2\{\phi, h\} = h$ for all $\{\phi, h\} \in X$. By [16, Theorem II], we have that for all $\{\phi, h\} \in X$ and $t \geqslant 0$,

$$T(t)\{\phi, h\} = S(t)\{\phi, h\} - \int_0^t S(t - s)\, A_2T(s)\{\phi, h\}\, ds.$$

Therefore, recalling (5.1) and Proposition 5.3, we have that for all $\{\phi, h\} \in X$ and $t \geqslant 0$,

$$\begin{aligned}
x(\phi, h)(t) &= P_2T(t)\{\phi, h\} \\
&= P_2S(t)\{\phi, h\} - P_2 \int_0^t S(t - s)\, A_2T(s)\{\phi, h\}\, ds \\
&= h - \int_0^t P_2A_2T(s)\{\phi, h\}\, ds \\
&= h + \int_0^t (f(P_2T(s)\{\phi, h\}) + F(P_1T(s)\{\phi, h\}))\, ds \\
&= h + \int_0^t (f(x(\phi, h)(s) + F(x_s(\phi, h)))\, ds.
\end{aligned}$$

The conclusion (5.13) follows from the continuity of f and F.

The differentiability results of Propositions 5.8 and 5.9 can be used to obtain some additional properties of the semigroup $T(t)$, $t \geqslant 0$.

PROPOSITION 5.10. *Suppose that F is as in Section 3. If f is continuous and everywhere defined, then $T(t)(D(A)) \subseteq D(A)$ for $t \geqslant 0$ and $T(t)(X) \subseteq D(A)$ for $t \geqslant r$. If $f = 0$, then $AT(t) \in \mathrm{Lip}(X, X)$ for $t \geqslant r$. If $f = 0$ and H is finite dimensional, then $T(t)$ is compact for $t \geqslant r$.*

Proof. If $\{\phi, h\} \in X$, then (5.13) yields that $x(\phi, h)(t)$ is continuously differentiable for $t \geqslant 0$. The first claim then follows by the definition of $D(A)$. Suppose $f = 0$ and let $\{\phi, h\}, \{\psi, k\} \in X$, and $t \geqslant r$. By (5.1), (5.13), and Proposition 5.3 we have

$$\begin{aligned}
&\| AT(t)\{\phi, h\} - AT(t)\{\psi, k\}\|^p \\
&= \int_{-r}^{0} |(P_1AT(t)\{\phi, h\})(\theta) - (P_1AT(t)\{\psi, k\})(\theta)|^p \, d\theta \\
&\quad + | P_2AT(t)\{\phi, h\} - P_2AT(t)\{\psi, k\}|^p \\
&= \int_{-r}^{0} | F(x_{t+\theta}(\phi, h)) - F(x_{t+\theta}(\psi, k))|^p \, d\theta \\
&\quad + F(x_t(\phi, h)) - F(x_t(\psi, k))|^p \\
&\leqslant \beta^p(r \sup_{\theta\in[-r,0]} \| x_{t+\theta}(\phi, h) - x_{t+\theta}(\psi, k)\|^p \\
&\quad + \| x_t(\phi, h) - x_t(\psi, k)\|^p \\
&= \beta^p(r \sup_{\theta\in[-r,0]} \| P_1T(t+\theta)\{\phi, h\} - P_1T(t+\theta)\{\psi, k\}\|^p \\
&\quad + \| P_1T(t)\{\phi, h\} - P_1T(t)\{\psi, k\}\|^p \\
&\leqslant \beta^p(re^{\gamma tp} + e^{\gamma tp}) \|\{\phi, h\} - \{\psi, k\}\|^p.
\end{aligned}$$

The second claim now follows. The third claim follows from the second claim, Proposition 3.2, and the fact that $T(t) = (I + \lambda A)^{-1}(I + \lambda A)\, T(t)$ for $t \geqslant r$ and λ positive and sufficiently small.

PROPOSITION 5.11. *Suppose that F is as in Section 4 and either $p > 1$ or F is linear. If f is continuous and everywhere defined, then $T(t)(D(A)) \subseteq D(A)$ for $t \geqslant 0$. If $f = 0$ then $T(t)(X) \subseteq D(A)$ for $t \geqslant r$ and $AT(t) \in \mathrm{Lip}(X, X)$ for $t \geqslant r$. If $f = 0$ and H is finite dimensional, then $T(t)$ is compact for $t \geqslant r$.*

Proof. If $\{\phi, h\} \in D(A)$, then $x(\phi, h)(t)$ is continuous for $t \geqslant -r$, and therefore for $t, s \geqslant 0$

$$|F(x_t(\phi, h)) - F(x_s(\phi, h))| \leqslant \beta\tau(0) \sup_{\theta\in[-r,0]} |x(\phi, h)(t+\theta) - x(\phi, h)(s+\theta)|.$$

Consequently, $x(\phi, h)(t)$ is continuously differentiable for $t \geqslant 0$ by (5.11), and thus $T(t)(D(A)) \subseteq D(A)$ for $t \geqslant 0$ when f is continuous and everywhere defined.

In order to prove the other claims, we will establish that for $f = 0$, $\{\phi, h\}, \{\psi, k\} \in D(A)$, $t \geqslant r$,

$$\int_{-r}^{0} |\dot{x}(\phi, h)(t+\theta) - \dot{x}(\psi, k)(t+\theta)|^p \tau(\theta)\, d\theta \leqslant \text{const}\, \|\{\phi, h\} - \{\psi, k\}\|^p. \tag{5.14}$$

For each positive integer n let $-r = \theta_0{}^n < \theta_1{}^n < \cdots < \theta_m{}^n = 0$, where $\theta_i{}^n - \theta_{i-1}^n < 1/n$. Then, for $t - r \leqslant \theta \leqslant t$,

$$\begin{aligned} w_n(\theta) &\overset{\text{def}}{=} \Bigg| g\left(\sum_{i=1}^{m} (\eta(\theta_i{}^n) - \eta(\theta_{i-1}^n))\, x(\phi, h)(t+\theta+\theta_i{}^n)\right) \\ &\quad - g\left(\sum_{i=1}^{m} (\eta(\theta_i{}^n) - \eta(\theta_{i-1}^n))\, x(\psi, k)(t+\theta+\theta_i{}^n)\right)\Bigg| \\ &\leqslant \beta \sum_{i=1}^{m} (\tau(\theta_i{}^n) - \tau(\theta_{i-1}^n))\, |x(\phi, h)(t+\theta+\theta_i{}^n) - x(\psi, k)(t+\theta+\theta_i{}^n)|. \end{aligned}$$

Then,

$$\begin{aligned} &\left(\int_{-r}^{0} w_n(\theta)^p \tau(\theta)\, d\theta\right)^{1/p} \\ &\leqslant \beta \left(\int_{-r}^{0} \left(\sum_{i=1}^{m} (\tau(\theta_i{}^n) - \tau(\theta_{i-1}^n))\, |x(\phi, h)(t+\theta+\theta_i{}^n) \right.\right. \\ &\qquad \left.\left. - x(\psi, k)(t+\theta+\theta_i{}^n)|\right)^p \tau(\theta)\, d\theta\right)^{1/p} \\ &\leqslant \beta \sum_{i=1}^{m} (\tau(\theta_i{}^n) - \tau(\theta_{i-1}^n)) \left(\int_{-r}^{0} |x(\phi, h)(t+\theta+\theta_i{}^n) \right. \\ &\qquad \left. - x(\psi, k)(t+\theta+\theta_i{}^n)|^p \tau(\theta)\, d\theta\right)^{1/p}. \end{aligned}$$

If $t + \theta_i^n \leqslant r$, then $-r \leqslant -t - \theta_i^n \leqslant 0$ and

$$\begin{aligned}
&\int_{-r}^{-t-\theta_i^n} | x(\phi, h)(t + \theta + \theta_i^n) - x(\psi, k)(t + \theta + \theta_i^n)|^p \tau(\theta)\, d\theta \\
&\qquad + \int_{-t-\theta_i^n}^{0} | x(\phi, h)(t + \theta + \theta_i^n) - x(\psi, k)(t + \theta + \theta_i^n)|^p \tau(\theta)\, d\theta \\
&\quad \leqslant \int_{t-r+\theta_i^n}^{0} | \phi(\theta) - \psi(\theta)|^p \tau(\theta)\, d\theta \\
&\qquad + \int_{-t-\theta_i^n}^{0} | P_2T(t + \theta + \theta_i^n)\{\phi, h\} - P_2T(t + \theta + \theta_i^n)\{\psi, k\}|^p \tau(\theta)\, d\theta \\
&\quad \leqslant (1 + \tau(0)\, re^{\gamma tp}) \|\{\phi, h\} - \{\psi, k\}\|^p.
\end{aligned}$$

If $t + \theta_i^n \geqslant r$, then

$$\begin{aligned}
&\int_{-r}^{0} | x(\phi, h)(t + \theta + \theta_i^n) - x(\psi, k)(t + \theta + \theta_i^n)|^p \tau(\theta)\, d\theta \\
&\qquad = \int_{-r}^{0} | P_2T(t + \theta + \theta_i^n)\{\phi, h\} - P_2T(t + \theta + \theta_i^n)\{\psi, k\}|^p \tau(\theta)\, d\theta \\
&\qquad \leqslant \tau(0)\, re^{\gamma tp} \|\{\phi, h\} - \{\psi, k\}\|^p.
\end{aligned}$$

Thus,

$$\left(\int_{-r}^{0} w_n(\theta)^p \tau(\theta)\, d\theta\right)^{1/p} \leqslant \beta\tau(0)(1 + \tau(0)\, re^{\gamma tp})^{1/p} \|\{\phi, h\} - \{\psi, k\}\|.$$

Since $w_n(\theta)$ converges to $| \dot{x}(\phi, h)(t + \theta) - \dot{x}(\psi, k)(t + \theta)|$ for all $\theta \in [-r, 0]$, Fatou's lemma yields (5.14).

Next, we observe that for $\{\phi, h\}, \{\psi, k\} \in D(A)$, $t \geqslant r$,

$$\begin{aligned}
&| F(x_t(\phi, h)) - F(x_t(\psi, k)) \\
&\qquad \leqslant \beta\tau(0) \sup_{\theta\in[-r,0]} | x(\phi, h)(t + \theta) - x(\psi, k)(t + \theta)| \\
&\qquad = \beta\tau(0) \sup_{\theta\in[-r,0]} | P_2T(t + \theta)\{\phi, h\} - P_2T(t + \theta)\{\psi, k\}| \\
&\qquad \leqslant \beta\tau(0)\, e^{\gamma t} \|\{\phi, h\} - \{\psi, k\}\|. \qquad (5.15)
\end{aligned}$$

Then, for $\{\phi, h\}, \{\psi, k\} \in D(A)$, $t \geqslant r$, (5.14) and (5.15) yield

$$\begin{aligned}
&\| AT(t)\{\phi, h\} - AT(t)\{\psi, k\}\|^p \\
&= \int_{-r}^{0} | d/d\theta(P_1T(t)\{\phi, h\})(\theta) \\
&\quad - d/d\theta(P_1T(t)\{\psi, k\})(\theta)|^p \tau(\theta)\, d\theta \\
&\quad + |F(P_1T(t)\{\phi, h\}) - F(P_1T(t)\{\psi, k\})|^p \\
&= \int_{-r}^{0} | \dot{x}(\phi, h)(t+\theta) - \dot{x}(\psi, k)(t+\theta)|^p \tau(\theta)\, d\theta \\
&\quad + |F(x_t(\phi, h)) - F(x_t(\psi, k))|^p \\
&\leqslant \text{const}\, \|\{\phi, h\} - \{\psi, k\}\|^p. \qquad (5.16)
\end{aligned}$$

Hence, $AT(t)$ is Lipschitz continuous on $D(A)$ for $t \geqslant r$.

Let λ be positive and sufficiently small such that $(I + \lambda A)^{-1} \in \operatorname{Lip}(X, X)$. Then, $(I + \lambda A)$ is closed, as is A. Let $t \geqslant r$, $\{\phi, h\} \in X$, and $\{\phi_n, h_n\} \subseteq D(A)$ such that $\lim_{n\to\infty}\{\phi_n, h_n\} = \{\phi, h\}$. Since $AT(t)$ is Lipschitz continuous on $D(A)$, $AT(t)\{\phi_n, h_n\}$ converges in X. By the closedness of A we must have

$$T(t)\{\phi, h\} \in D(A) \qquad \text{and} \qquad \lim_{n\to\infty} AT(t)\{\phi_n, h_n\} = AT(t)\{\phi, h\}. \quad (5.17)$$

Thus, $T(t)(X) \subseteq D(A)$ for $t \geqslant r$. Also, (5.16) and (5.17) imply that $AT(t) \in \operatorname{Lip}(X, X)$ for $t \geqslant r$. The compactness of $T(t)$ for $t \geqslant r$ when H is finite dimensional then follows from Proposition 4.2 and the identity $T(t) = (I + \lambda A)^{-1}(I + \lambda A)\, T(t)$, thus completing the proof.

If F is as in Section 4, then Proposition 5.8 can be used to obtain additional information about the sense in which the trajectories of $T(t)$, $t \geqslant 0$ satisfy (FDE) for $\{\phi, h\} \notin D(A)$.

PROPOSITION 5.12. *Suppose F is as in Section 4, $f = 0$, and either $p > 1$ or F is linear. If $\{\phi, h\} \in X$ and $\{\phi_n, h_n\} \subseteq D(A)$ such that $\lim_{n\to\infty}\{\phi_n, h_n\} = \{\phi, h\}$, then*

$$x(\phi, h)(t) = h + \lim_{n\to\infty} \int_0^t F(x_s(\phi_n, h_n))\, ds \qquad \textit{for} \quad t \geqslant 0. \quad (5.18)$$

Proof. From (5.11) and (5.14), we have that for $t \leqslant kr$, k a positive integer, and $K = \lim_{\theta \to -r} \tau(\theta)$,

$$\int_0^t |F(x_s(\phi_n, h_n)) - F(x_s(\phi_m, h_m))| \, ds$$

$$= \int_0^t | \dot{x}(\phi_n, h_n)(s) - \dot{x}(\phi_m, h_m)(s)| \, ds$$

$$\leqslant \sum_{i=1}^{k} \int_{-r}^{0} | \dot{x}(\phi_n, h_n)(ir + \theta) - \dot{x}(\phi_m, h_m)(ir + \theta)| \, d\theta$$

$$\leqslant K^{-1} \sum_{i=1}^{k} \int_{-r}^{0} | \dot{x}(\phi_n, h_n)(ir + \theta) - \dot{x}(\phi_m, h_m)(ir + \theta)| \, \tau(\theta) \, d\theta$$

$$\leqslant K^{-1} \sum_{i=1}^{k} \left(\int_{-r}^{0} \tau(\theta) \, d\theta\right)^{1/q} \left(\int_{-r}^{0} | \dot{x}(\phi_n, h_n)(ir + \theta) - \dot{x}(\phi_m, h_m)(ir + \theta)|^p \, \tau(\theta) \, d\theta\right)^{1/p}$$

$$\leqslant \text{const} \, \|\{\phi_n, h_n\} - \{\phi_m, h_m\}\|.$$

Thus, $\lim_{n\to\infty} \int_0^t F(x_s(\phi_n, h_n)) \, ds$ exists for $t \geqslant 0$. Also, for $t \geqslant 0$,

$$| x(\phi, h)(t) - x(\phi_n, h_n)(t)| = | P_2 T(t)\{\phi, h\} - P_2 T(t)\{\phi_n, h_n\}|$$
$$\leqslant \| T(t)\{\phi, h\} - T(t)\{\phi_n, h_n\}\| \leqslant e^{\gamma t} \|\{\phi, h\} - \{\phi_n, h_n\}\|.$$

Then, (5.18) follows from (5.11).

COROLLARY 5.13. *Suppose $f = 0$, F is as in Section 4, $g = I$, and $\eta(\theta)$ is linear for each $\theta \in [-r, 0]$. Then, for $t \geqslant 0$, $\{\phi, h\} \in X$,*

$$x(\phi, h)(t) = h + \int_{-r}^{0} d\eta(\theta) \left(\int_0^t x(\phi, h)(s + \theta) \, ds\right). \tag{5.19}$$

Proof. If $\{\phi, h\} \in X$ and $\theta \in [-r, 0]$, then $\int_0^t x(\phi, h)(s + \theta) \, ds$ is continuous in t. Also, if $\{\phi_n, h_n\} \subseteq D(A)$ such that $\lim_{n\to\infty}\{\phi_n, h_n\} = \{\phi, h\}$ and $\theta \in [-r, 0]$, then

$$\left| \int_0^t (x(\phi_n, h_n)(s + \theta) - x(\phi, h)(s + \theta)) \, ds \right|$$
$$\leqslant \text{const} \, \|\{\phi_n, h_n\} - \{\phi, h\}\|.$$

Since

$$\int_0^t \left(\int_{-r}^0 d\eta(\theta)\, x(\phi_n, h_n)(s+\theta)\right) ds = \int_{-r}^0 d\eta(\theta) \left(\int_0^t x(\phi_n, h_n)(s+\theta)\, ds\right)$$

and

$$\lim_{n\to\infty} \int_{-r}^0 d\eta(\theta) \left(\int_0^t x(\phi_n, h_n)(s+\theta)\, ds\right)$$
$$= \int_{-r}^0 d\eta(\theta) \left(\int_0^t x(\phi, h)(s+\theta)\, ds\right),$$

(5.19) follows from Proposition 5.12.

COROLLARY 5.14. *Suppose* $f = 0$, F *is as in Section* 4, $F(\phi) = g(\sum_{i=1}^m \eta_i \phi(-r_i))$, *where* $\eta_i \in \mathrm{Lip}(H, H)$ *and* $0 < r_i \leqslant r$, *and* $p > 1$. *Then, for* $t \geqslant 0$, $\{\phi, h\} \in X$,

$$x(\phi, h)(t) = h + \int_0^t g\left(\sum_{i=1}^m \eta_i(x(\phi, h)(s - r_i))\right) ds. \tag{5.20}$$

Proof. The proof follows from the fact that for $\{\phi_n, h_n\} \in D(A)$,

$$\int_0^t F(x_s(\phi_n, h_n))\, ds = \int_0^t g\left(\sum_{i=1}^m \eta_i(x(\phi_n, h_n)(s - r_i))\right) ds.$$

As in Corollary 5.13, if $\lim_{n\to\infty}\{\phi_n, h_n\} = \{\phi, h\}$, then

$$\lim_{n\to\infty} \int_0^t g\left(\sum_{i=1}^m \eta_i(x(\phi_n, h_n)(s - r_i))\right) ds$$
$$= \int_0^t g\left(\sum_{i=1}^m \eta_i(x(\phi, h)(s - r_i))\right) ds.$$

The conclusion then follows from Proposition 5.12.

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